The Continuous Time Nonzero-sum Dynkin Game and Applications.

S. Hamadène
Univ. of Le Mans (Fr.)

(Joint work with J. Zhang, USC, L.A. (USA), published in SIAM JCO, 2010.)

Outlines

1. Introduction

2. The particular case of zerosum game

3. The state of the art

4. The main result

5. Application in game options.
1. Introduction

- $(\Omega, \mathcal{F}, (F_t)_{t \leq T}, P)$ is a filtered probability space with $\mathcal{F} = (F_t)_{t \leq T}$ complete and right continuous which is not necessarily generated by a BM; $T$ is the horizon of the problem.

- Two players $a_1$ and $a_2$ act on a system up to the time when one of them decides to stop controlling, at an $\mathcal{F}$-stopping time $\tau_1$ (resp. $\tau_2$) for $a_1$ (resp. $a_2$).

- The reward for $a_1$ (resp. $a_2$) is given by

$$J_1(\tau_1, \tau_2) \triangleq E\left\{ X^1_{\tau_1} 1\{\tau_1 \leq \tau_2\} + Y^1_{\tau_2} 1\{\tau_2 < \tau_1\} \right\}$$

(resp.)

$$J_2(\tau_1, \tau_2) \triangleq E\left\{ X^2_{\tau_2} 1\{\tau_2 < \tau_1\} + Y^2_{\tau_1} 1\{\tau_1 \leq \tau_2\} \right\}$$

where the processes $X^i, Y^i, i = 1, 2$, are $\mathcal{F}$-adapted and of class $[D]$. 

3
**Definition:** A pair \((\tau_1^*, \tau_2^*)\) of \(F\)-stopping times is called a Nash equilibrium point for the NZSDG if it satisfies: \(\forall \tau_1, \tau_2 \in \mathcal{T}_0,\)

\[
J_1(\tau_1, \tau_2^*) \leq J_1(\tau_1^*, \tau_2^*)
\]

and

\[
J_2(\tau_1^*, \tau_2) \leq J_2(\tau_1^*, \tau_2^*).
\]

In discrete time framework, the problem is already discussed (Morimoto, '84,'86).
2. Particular case of zerosum game

When $J_1 + J_2 = 0$ this corresponds to the zerosum Dynkin game (recallable options, convertible bonds,...) and a NEP for the game is just a saddle point for the ZSDG. It satisfies: $\forall \tau_1, \tau_2,$

$$J_1(\tau_1^*, \tau_2) \leq J_1(\tau_1^*, \tau_2^*) \leq J_1(\tau_1, \tau_2^*).$$

Connection with reflected BSDEs: In the case when $\mathbf{F}$ is generated either by a BM (or a BM and an independent Poisson random measure), the zerosum Dynkin problem is connected with Reflected BSDEs with two obstacles ($X^i \leq Y^i$).
Actually let \((Y_t, Z_t, K_t^\pm)_{t \leq T}\) be \(\mathbf{F}\)-adapted stochastic processes s.t. \(\forall t \leq T\),

\[
\begin{align*}
\bullet Y_t &= X^1_t + (K_T^+ - K_t^+) - (K_T^- - K_t^-) - \int_t^T Z_s dB_s \\
\bullet X^1_t &\leq Y_t \leq Y^1_t \\
\bullet (Y_t - X^1_t) dK_t^+ &= (Y_t - Y^1_t) dK_t^- = 0
\end{align*}
\]

then

\[
Y_0 = \text{essinf}_{\tau_2} \text{esssup}_{\tau_1} J^1(\tau_1, \tau_2) = \text{esssup}_{\tau_1} \text{essinf}_{\tau_2} J^1(\tau_1, \tau_2)
\]

and

\[
\tau_1^* = \inf\{s \geq 0, Y_t = X^1_t\},
\]

\[
\tau_2^* = \inf\{s \geq 0, Y_t = Y^1_t\} \wedge T
\]

is a saddle-point for the zero-sum game.
3. State of the art in the NZSDG case

The streamline is the following result:

Proposition: Assume there exist two continuous $F$-supermartingales $(W^i_t)_{t \leq T}$, $i = 1, 2$, and two stopping times $\tau^*_i$, $i = 1, 2$, such that:

(i) $W^i \geq X^i$, for $i = 1, 2$

(ii) $W^1_{\tau^*_1} = X^1_{\tau^*_1}$ on $\tau^*_1 \leq \tau^*_2$ and

$\quad W^2_{\tau^*_2} = X^2_{\tau^*_2}$ on $\tau^*_2 < \tau^*_1$

(iii) $W^1_{\tau^*_2} = Y^1_{\tau^*_2}$ and $W^2_{\tau^*_1} = Y^2_{\tau^*_1}$

(iv) $W^2_{t \wedge \tau^*_1}$ and $W^1_{t \wedge \tau^*_2}$ are supermartingales

(v) $W^i_{t \wedge \tau^*_1 \wedge \tau^*_2}$, $i = 1, 2$, are martingales.

Then the pair $(\tau^*_1, \tau^*_2)$ is a NEP for the game.
Sketch of the proof:

(i)

\[
E[W^1_0] = E[W^1_{\tau_1^* \land \tau_2^*}] = E[W^1_{\tau_1^* 1[\tau_1^* \leq \tau_2^*]} + W^1_{\tau_2^* 1[\tau_2^* < \tau_1^*]}]
= E[X^1_{\tau_1^* 1[\tau_1^* \leq \tau_2^*]} + Y^1_{\tau_2^* 1[\tau_2^* < \tau_1^*]}]
= J^1(\tau_1^*, \tau_2^*).
\]

(ii) Let \(\tau_1\) be a stopping time, then:

\[
E[W^1_0] \geq E[W^1_{\tau_1^* \land \tau_2^*}] = E[W^1_{\tau_1^* 1[\tau_1^* \leq \tau_2^*]} + W^1_{\tau_2^* 1[\tau_2^* < \tau_1^*]}] \geq E[X^1_{\tau_1^* 1[\tau_1^* \leq \tau_2^*]} + Y^1_{\tau_2^* 1[\tau_2^* < \tau_1^*]}] = J^1(\tau_1, \tau_2^*).
\]

Thus

\[
E[W^1_0] = J^1(\tau_1^*, \tau_2^*) \geq J^1(\tau_1, \tau_2^*).
\]

In the same way we have

\[
E[W^2_0] = J^2(\tau_1^*, \tau_2^*) \geq J^2(\tau_1^*, \tau_2)
\]

for any \(\tau_2\) s.t.. Then \((\tau_1^*, \tau_2^*)\) is a NEP for the NZSDG.
A. PDE approach (Bens.-Fried., ’77)

Assume:

- $\zeta := (\zeta_t)_{t \leq T}$ is a solution of a standard differential equation i.e.

\[
\begin{align*}
  d\zeta_t &= b(t, \zeta_t) dt + \sigma(t, \zeta_t) dB_t, \quad t \leq T; \\
  \zeta_0 &= x \in \mathbb{R}^k
\end{align*}
\]

whose infinitesimal generator is $A$ i.e.

\[
A \Phi(t, x) = \frac{1}{2} \sum_{i,j=1,k} (\sigma \sigma^T)(t, x) \partial_{ij} \Phi(t, x) + \sum_{i=1,k} b_i(t, x) \partial_i \Phi(t, x).
\]

- $X^i_t = \varphi^i(t, \zeta_t)$ and $Y^i = \psi^i(t, \zeta_t)$ where $\psi^i$ and $\varphi^i$ deterministic continuous functions

- [H1]: $X^i \leq Y^i$ and $X^i_T = Y^i_T$

- [H2]: $Y^i$ supermartingales.
**Theorem (B-F, ’77):** There exist two deterministic continuous bounded functions $u^1(t,x)$ and $u^2(t,x)$ solution of the following system:

\[
\begin{align*}
    u^i(T, x) &= \psi^i(T, x); \\
    u^i &\geq \phi^i; \\
    \text{if } u^j(t,x) &= \phi^j(t,x) \text{ for } j \neq i \\
    \text{and some } (t,x), \text{ then } u^i(t,x) &= \psi^i(t,x); \\
    \text{if } \Sigma^i = \{(t,x), u^j(t,x) > \phi^j(t,x) \text{ for } j \neq i\}, \\
    \text{then } Au^i(t,x) &\geq 0 \text{ for } (t,x) \in \Sigma^i; \\
    (u^i - \phi^i).Au^i(t,x) &= 0 \text{ in } \Sigma^i
\end{align*}
\]

(1)

and the following pair of stopping times,

\[
\hat{\tau}_i = \inf\{s \geq 0, u^i(s, \zeta_s) = \varphi^i(s, \zeta_s)\} \wedge T; i = 1, 2
\]

is a NEP for the NZSDG.
B. The probabilistic approach (Etourneau, 86)

*Theorem*: The processes are general and satisfy [H1]-[H2]. Then the NZSDG has a NEP.

The proof uses the notion of **Snell envelope** of processes which is the following:

Let $U$ be an RCLL adapted stochastic process. The Snell envelope of $U$, denoted by $R(U)$, is the smallest $\mathcal{F}_t$-supermartingale which dominates $U$, i.e., if $\bar{W}$ is another RCLL supermartingale such that $\bar{W}_t \geq U_t$ for all $0 \leq t \leq T$, then $\bar{W}_t \geq W_t$ for any $0 \leq t \leq T$. 
The continuous case.
rcII with positive jumps
RCLL with negative jumps
It satisfies the following properties:

\( \text{(i) For any } F\text{-stopping time } \theta \text{ we have:} \)

\[ W_\theta = \text{esssup}_{\tau \geq \theta} E[U_\tau | \mathcal{F}_\theta] \; P - a.s.(W_T = U_T); \]
(ii) Assume that $U$ has only positive jumps. Then the stopping time

$$\tau^* \triangleq \inf\{s \geq 0, W_s = U_s\} \wedge T$$

is optimal, i.e.,

$$E[W_0] = E[W_{\tau^*}] = E[U_{\tau^*}] = \sup_{\tau \in \mathcal{T}_0} E[U_\tau].$$

As a by-product we have $W_{\tau^*} = U_{\tau^*}$ and the process $W$ is a martingale on the time interval $[0, \tau^*]$. ■
The main idea of Etourneau’s proof is:

Let $\mathcal{E}_1$ (resp. $\mathcal{E}_2$) be the set of RCLL $\mathbf{F}$-adapted processes $V^1$ (resp. $V^2$) s.t. $X^1 \leq V^1 \leq Y^1$ (resp. $X^2 \leq V^2 \leq Y^2$).

For $(i,j) = (1,2)$ (resp. $(2,1)$) and for $V^j \in \mathcal{E}_j$

$$D_j = \inf\{s \geq 0, V^j_s = X^j_s\} \wedge T$$

and

$$f_i(V^j) = R(X^i 1_{[0,D_j]} + Y^i 1_{[D_j,T]}).$$

Then:

$f_i$ is a decreasing map from $\mathcal{E}_j$ to $\mathcal{E}_i$.

Therefore the mappings $f_1 o f_2$ (resp. $f_2 o f_1$) are non-decreasing of $\mathcal{E}_1$ (resp. $\mathcal{E}_2$) and have fixed points $W^1$ and $W^2$ which provide a NEP for the NZSDG whose expression is:
\[ \tau_1^* = \inf\{s \geq 0, W_s^1 = X_s^1\} \wedge T \]

and

\[ \tau_2^* = \inf\{s \geq 0, W_s^2 = X_s^2\} \wedge T. \]
4. **The main result**: without [H2].

**Theorem** (Ham.-J. Zhang, ’10): Assume:

- [H1] i.e. $X^1 \leq Y^1$, $X^2 \leq Y^2$ and $X^1_T = Y^1_T$ (technical and can be removed); those processes are RCLL and $X^i$, $i = 1, 2$, have only positive jumps.

- for any stopping time $\tau$,

\[ P[\{X^1_\tau < Y^1_\tau\} - \{X^2_\tau < Y^2_\tau\}] = 0 \]

(assumption which is satisfied if $X^2 < Y^2$).

Then the NZSDG has a NEP $(\tau^*_1, \tau^*_2)$. 
Sketch of the proof:

Let \( \tau_1 = T \) and \( \tau_2 = T \). For \( n \geq 1 \), assume \( \tau_{2n-1} \) and \( \tau_{2n} \) defined, we then define \( \tau_{2n+1} \) and \( \tau_{2n+2} \) as: Let

\[
W_{t}^{2n+1} = \underset{\tau \geq t}{\text{esssup}} E[X_{\tau}^{1} 1_{\{\tau < \tau_{2n}\}} + Y_{\tau_{2n}}^{1} 1_{\{\tau \geq \tau_{2n}\}|F_{t}}]
\]

\[
\tilde{\tau}_{2n+1} = \inf\{t \geq 0 : W_{t}^{2n+1} = X_{t}^{1}\} \land \tau_{2n}
\]

and

\[
\tau_{2n+1} = \begin{cases} 
\tilde{\tau}_{2n+1}, & \text{if } \tilde{\tau}_{2n+1} < \tau_{2n}; \\
\tau_{2n-1}, & \text{if } \tilde{\tau}_{2n+1} = \tau_{2n}.
\end{cases}
\]
Next, let

\[ W_t^{2n+2} = \text{esssup}_{\tau \geq t} E[X_{\tau}^2 1_{\{\tau < \tau_{2n+1}\}} + Y_{\tau_{2n+1}}^2 1_{\{\tau \geq \tau_{2n+1}\}}|F_t], \]

\[ \tilde{\tau}_{2n+2} = \inf\{t \geq 0 : W_t^{2n+2} = X_t^2\} \wedge \tau_{2n+1} \]

and

\[ \tau_{2n+2} = \begin{cases} 
\tilde{\tau}_{2n+2}, & \text{if } \tilde{\tau}_{2n+2} < \tau_{2n+1}; \\
\tau_{2n}, & \text{if } \tilde{\tau}_{2n+2} = \tau_{2n+1}. 
\end{cases} \]

The sequences \((\tau_{2n})_{n \geq 0}\) and \((\tau_{2n+1})_{n \geq 0}\) are decreasing and converge respectively to \(\tau^*_1\) and \(\tau^*_2\) respectively and \((\tau^*_1, \tau^*_2)\) is a NEP for the NZSDG.
Step 1: for any stopping time $\tau$,

$$J_1(\tau, \tau_{2n}) \leq J_1(\tau_{2n+1}, \tau_{2n})$$

and

$$J_2(\tau_{2n+1}, \tau) \leq J_2(\tau_{2n+1}, \tau_{2n+2}).$$

By definition of $W^{2n+1}$,

- $W_{\tau_{2n}}^{2n+1} = Y_{\tau_{2n}}^1$

- $W_t^{2n+1} \geq X_t^1$ for any $t \in [0, \tau_{2n}]$

- $W^{2n+1}$ is a supermartingale over $[0, \tau_{2n}]$ and a martingale over $[0, \tilde{\tau}_{2n}]$

- On $\{\tau_n = \tau_{n-1}\}$ we have $\tau_m = T$ for $m \leq n$. 

23
Then, for any $\tau$,

\[
J_1(\tau, \tau_{2n}) = E\left\{ X_1^1 \mathbb{1}_{\tau \leq \tau_{2n}} + Y_{\tau_{2n}}^1 \mathbb{1}_{\tau_{2n} < \tau} \right\}
\leq E\left\{ W_{\tau}^{2n+1} \mathbb{1}_{\tau \leq \tau_{2n}} + W_{\tau_{2n}}^{2n+1} \mathbb{1}_{\tau_{2n} < \tau} \right\}
= E\{ W_{\tau_{2n} \wedge \tau}^{2n+1} \} \leq W_0^{2n+1}.
\]

But

\[
J_1(\tau_{2n+1}, \tau_{2n}) = \\
E\left\{ X_{\tau_{2n+1}}^1 \mathbb{1}_{\tau_{2n+1} \leq \tau_{2n}} + Y_{\tau_{2n}}^1 \mathbb{1}_{\tau_{2n} < \tau_{2n+1}} \right\}
= E\left\{ X_{\tau_{2n+1}}^1 \mathbb{1}_{\tau_{2n+1} < \tau_{2n}} + Y_{\tau_{2n}}^1 \mathbb{1}_{\tau_{2n} \leq \tau_{2n+1}} \right\}.
\]

Then

\[
J_1(\tau_{2n+1}, \tau_{2n}) = \\
E\left\{ X_{\tilde{\tau}_{2n+1}}^1 \mathbb{1}_{\tilde{\tau}_{2n+1} < \tau_{2n}} + W_{\tau_{2n}}^{2n+1} \mathbb{1}_{\tilde{\tau}_{2n+1} = \tau_{2n}} \right\}
= E\{ W_{\tilde{\tau}_{2n+1}}^{2n+1} \} = W_0^{2n+1}.
\]

Therefore

\[
J_1(\tau, \tau_{2n}) \leq J_1(\tau_{2n+1}, \tau_{2n}).
\]

In the same way we have the other inequality.
Step 2:

(i) for any s.t. $\tau$, $\lim_n J_1(\tau, \tau_{2n}) = J_1(\tau, \tau^*_2)$

(ii) for any s.t. $\tau$ satisfying $P[\tau = \tau^*_1 < T] = 0$, $\lim_n J_2(\tau_{2n+1}, \tau) = J_2(\tau^*_1, \tau)$.

Recall

$$J_1(\tau, \tau_{2n}) = E\left\{ X^1 \tau_1 1[\tau \leq \tau_{2n}] + Y^1 \tau_{2n} 1[\tau_{2n} < \tau] \right\}$$

The result follows from RCLL of the processes, $\tau_n \geq \tau_{n+1}$ and the limit.

(ii) For the other limit one requires an additional assumption due to lack of continuity.
Step 3:

\[
\lim_{n} J_1(\tau_{2n+1}, \tau_{2n}) = J_1(\tau^*_1, \tau^*_2)
\]

and

\[
\lim_{n} J_2(\tau_{2n+1}, \tau_{2n+2}) = J_2(\tau^*_1, \tau^*_2).
\]

Step 4:

(i) Taking the limit above we have:

\[
J_1(\tau, \tau^*_2) \leq J_1(\tau^*_1, \tau^*_2).
\]

(ii) Let \( \tau \) be a stop. time and define

\[
\hat{\tau}_n = \{(\tau + \frac{1}{n}) \land T\}1\{[\tau = \tau^*_1 < T]:=A\} + \tau 1_{\sim A}.
\]

Then \( P[\hat{\tau}_n = \tau^*_1 < T] = 0 \) and

\[
J_2(\tau^*_1, \hat{\tau}_n) \leq J_2(\tau^*_1, \tau^*_2).
\]

Finally take the limit as \( n \to \infty \) to obtain the desired result.
4. Application in game options

Assume we have an American game contingent claim whose payoff is:

\[ \Gamma(\tau, \sigma) = L_\sigma 1_{[\sigma \leq \tau, \sigma < T]} + U_\tau 1_{[\tau < \sigma]} + \xi 1_{[\tau = \sigma = T]} \cdot \]

- \( L \leq U \) and the difference \( U - L \) is the compensation that \( a_1 \) pays to \( a_2 \) for the decision to terminate the contract before maturity date \( T \).

In a complete market the value of the GCC is given by:

\[ V_0 = \sup_{\sigma \geq 0} \inf_{\tau \geq 0} E^*[\Gamma(\tau, \sigma)] \]

\[ = \inf_{\tau \geq 0} \sup_{\sigma \geq 0} E^*[\Gamma(\tau, \sigma)]. \]

In incomplete markets another point of view is related to utility maximization of the agents (Kuhn, 03).
Let $\varphi_1, \varphi_2 : \mathbb{R} \to \mathbb{R}$ be two utility functions of the seller, respectively, the buyer of the GCC. The seller $a_1$ (resp. the buyer $a_2$) chooses a stopping time $\tau$ (resp. $\sigma$) in order to maximize

$$J_1(\tau, \sigma) := E[\varphi_1(-\Gamma(\tau, \sigma))]$$

(resp.

$$J_2(\tau, \sigma) := E[\varphi_2(\Gamma(\tau, \sigma))].$$

Therefore if the NZSDG associated with $J_1$ and $J_2$ has a NEP point $(\sigma^*, \tau^*)$, i.e.,

$$J_1(\tau^*, \sigma^*) \geq J_1(\tau, \sigma^*) \text{ and } J_2(\tau^*, \sigma^*) \geq J_2(\tau^*, \sigma)$$
then $-\varphi_1^{-1}(J_1(\tau^*, \sigma^*))$ (resp. $\varphi_2^{-1}(J_2(\tau^*, \sigma^*))$) is a seller (resp. buyer) price of the GCC.

**Theorem**: Assume that:

(i) The utility functions $\varphi_1$ and $\varphi_2$ are non-decreasing;

(ii) $L, U$ are continuous and $L_t \leq U_t$ and $L_T \leq \xi \leq U_T$, P-a.s.;

Then the nonzero-sum Dynkin game associated with the GCC has a Nash equilibrium point $(\tau^*, \sigma^*)$. 

29